Int. J. Solids Structures, 1976, Vol. 12, pp. 81–97. Pergamon Press. Printed in Great Britain

ELASTIC MODULI OF A CRACKED SOLID†

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(*Received* 10 *March* 1975; *revised* 27 *June* 1975)

Abstract—Calculations on the basis of the self-consistent method are made for the elastic moduli of bodies containing randomly distributed flat cracks, with or without fluid in their interiors. General concepts are outlined for arbitrary cracks and explicit derivations together with numerical results are given for elliptic cracks. Parameters are identified which adapt the elliptic-crack results to arbitrary convex crack shapes. Finally, some geometrical relations involving randomly distributed cracks and their traces on cross-sections are presented.

INTRODUCTION

The problem studied in this paper is the analytic estimation of the effective elastic moduli of a body permeated by many flat cracks. The uncracked material is assumed to be locally isotropic, and the statistical distributions of the sizes, shapes, locations and orientations of the cracks are supposed to be sufficiently random and uncorrelated as to render the cracked body isotropic and homogeneous in the large. Crack closure effects are ignored; that is, the cracks are assumed to have very small openings between their opposite faces, and the crack edges are considered to be blunt, so that sufficiently small stresses do not produce contact between the crack faces. The macroscopic incremental stress-strain relation for the cracked body will then be linear.

Earlier studies of this problem have been based on the assumption of circular cracks [1–3]‡, or long rectangular cracks [1], and were explicitly limited to dilute concentrations of cracks, sufficiently far apart to permit neglect of elastic interaction effects between cracks. In the present paper, cracks of general elliptic planform are considered, and, more importantly, the calculations are made on the basis of a self-consistent approach that seeks to take account, albeit approximately, of the influence of crack interaction. The self-consistent method has been exploited earlier in analogous analyses of the elastic properties of composite materials [4–6]. In its present application, use is made of the elastic solution for an isolated elliptic crack in an infinite medium, the pertinent features of which are given a succinct rederivation in this paper.

Calculations are also made in this paper for bodies containing fluid-filled cracks. Also, in an Appendix to the paper, it is shown how the relevant geometric parameter describing the crack density can be related to measurements of crack traces on a plane cross-section of the cracked body.

Comparison of the results for effective moduli with laboratory measurements, and discussion of their geophysical implications concerning earthquake prediction, are given elsewhere [7].

SELF-CONSISTENT PROCEDURE

The self-consistent procedure will be described for the case of empty ("dry") flat cracks of arbitrary shape. Consider first an uncracked, homogeneous, isotropic body in a state of uniform hydrostatic stress p maintained by prescribed boundary tractions. The potential energy of the body and its loads is then $\phi = -(p^2V/2K)$, where K is the bulk modulus of the material and V is the total volume. Now suppose that, with the external loading unchanged, the introduction of the random set of cracks under consideration produces a potential energy change $\Delta\phi$; the effective

†This work was supported in part by the National Science Foundation under grants NSF GP 34723 and NSF DMR 7203020, the Committee for Experimental Geology and Geophysics, and the Division of Engineering and Applied Physics, Harvard University.

‡We thank J. D. Eshelby for bringing to our attention the early work by Bristow in [1]. We have also just learned of the recent study by Salganik [18] of the effects of dilute concentrations of dry elliptic cracks.

bulk modulus $\bar{K}$ of the cracked body may then be defined by the relation

$$-\frac{p^2V}{2\bar{K}} = -\frac{p^2V}{2K} + \Delta\phi. \tag{1}$$

The key step in the self-consistent method of estimating $\Delta\phi$ is to calculate the energy loss produced by a single isolated crack in an infinite medium having the effective properties of the cracked body. By dimensional analysis this energy loss must have the form

$$\mathscr{E} = \frac{p^2a^3}{\bar{E}} f(\bar{\nu}) \tag{2}$$

where a is a characteristic linear crack dimension; $\bar{E}$ is the effective Young's modulus of the cracked body; $\bar{\nu}$ is its effective Poisson's ratio; and f is a non-dimensional shape factor, that can depend on $\bar{\nu}$ as well as on the crack shape. This quantity $\mathscr{E}$, the energy *released* from the body-load system by introduction of the crack, will, for convenience, be called the *crack energy*. The energy change (the *negative* of the sum of the crack energies) is

$$\Delta\phi = -\frac{p^2}{\bar{E}} \sum a^3 f(\bar{\nu}) \tag{3}$$

and substitution into (1) yields

$$\bar{K}/K = 1 - \frac{2N\langle a^3 f(\bar{\nu})\rangle}{3(1-2\bar{\nu})} \tag{4}$$

where N is the number of cracks per unit volume, and the angle brackets denote an average. In arriving at (4), the standard relation

$$\bar{E}/\bar{K} = 3(1-2\bar{\nu}) \tag{5}$$

was used. If crack size and shape are uncorrelated, (4) can be replaced by

$$\bar{K}/K = 1 - \frac{2N\langle a^3\rangle\langle f(\bar{\nu})\rangle}{3(1-2\bar{\nu})}. \tag{6}$$

A similar calculation can be made for the case of uniaxial tension s applied to the cracked body. This time, the effective Young's modulus $\bar{E}$ is defined by

$$-\frac{s^2V}{2\bar{E}} = -\frac{s^2V}{2E} + \Delta\phi. \tag{7}$$

Consider now the crack energy associated with an isolated crack having an orientation defined by the unit vector **m** normal to its plane and the unit vector **t** of some characteristic direction in its plane. Only the resolved stresses σ and τ normal and tangential to the plane of the crack can influence the crack energy, which must be a quadratic function of these stresses. By symmetry, the effects of σ and τ are uncoupled in an isotropic medium, and so

$$\mathscr{E} = \frac{a^3}{\bar{E}}[\sigma^2 f(\bar{\nu}) + \tau^2 g(\bar{\nu}, \beta)]. \tag{8}$$

Here $g(\bar{\nu}, \beta)$ is another non-dimensional shape factor, that depends also on the angle β between the resolved shear stress vector and the characteristic crack direction **t**. Using (see Fig. 1) $\sigma = s\cos^2\alpha$ and $\tau = s\sin\alpha\cos\alpha$, summing (8) over all cracks to get $-\Delta\phi$, and substituting into (7) gives

$$\bar{E}/E = 1 - 2N\langle a^3(f(\bar{\nu})\cos^4\alpha + g(\bar{\nu},\beta)\sin^2\alpha\cos^2\alpha)\rangle \tag{9}$$

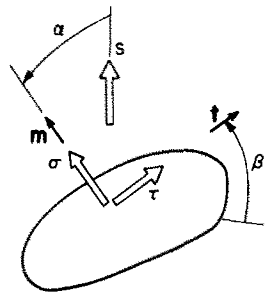

Fig. 1. Plane crack and resolved stresses.

With the further assumption that crack sizes, shapes, and orientations are uncorrelated, and the use of $\langle\cos^4\alpha\rangle = 1/5$, $\langle\sin^2\alpha\cos^2\alpha\rangle = 2/15$, this reduces to

$$\bar{E}/E = 1 - \frac{2N\langle a^3\rangle}{15}[3\langle f(\bar{\nu})\rangle + 2\langle g(\bar{\nu}, \beta)\rangle]. \tag{10}$$

The average of $g(\bar{\nu}, \beta)$ is with respect to crack shape, and, separately, with respect to β over the range $(0, \pi)$.

Equations (6) and (10), together with the relation (5), provide simultaneous equations for the determination of $\bar{K}$, $\bar{E}$, and $\bar{\nu}$, and also any other related elastic constants (such as the shear modulus $\bar{G}$). We must first, however, evaluate the crack energy terms in (8)—and hence $f(\bar{\nu})$ and $g(\bar{\nu}, \beta)$—for particular crack shapes.

CRACK ENERGIES RELATED TO STRESS INTENSITIES

Consider a flat crack of arbitrary shape lying within an isotropic elastic body subjected to an external surface load. Let s be the distance along the crack edge C (Fig. 2), let r be the normal outward distance from C to a point in the plane of the crack, and let z be the coordinate normal to the plane of the crack, so that (r, s, z) constitutes a right-handed coordinate system. Sufficiently near the crack boundary, the state of stress and strain must be essentially a combination of plane strain and antiplane shear. That is, for z, $r \to 0$, it must be true, asymptotically, that the stresses σ_z, σ_r, and τ_{rz}, together with the corresponding strains, satisfy the equations of plane strain; and the shear stresses τ_{rs}, τ_{zs}, independently, obey the equations of antiplane shear.† Accordingly, as in plane strain and antiplane shear, the stresses will have square-root singularities along the crack boundary, and the conventional stress-intensity factors

$$\left.\begin{aligned} K_{\rm I} &= \lim_{r\to 0}\sqrt{2\pi r}\,\sigma_z(r, s, 0) \\ K_{\rm II} &= \lim_{r\to 0}\sqrt{2\pi r}\,\tau_{zr}(r, s, 0) \\ K_{\rm III} &= \lim_{r\to 0}\sqrt{2\pi r}\,\tau_{zs}(r, s, 0) \end{aligned}\right\} \tag{11}$$

may be defined along C.

We recall next that in their study of so-called conservation laws of elastostatics, Knowles and Sternberg[9] show that the surface integral

$$M = \int_S \int \left\{W\mathbf{x}\,.\,\mathbf{n} - [(\mathbf{x}\,.\,\nabla)\mathbf{U}]\,.\,\mathbf{T} - \frac{1}{2}\mathbf{T}\,.\,\mathbf{U}\right\} \mathrm{d}S \tag{12}$$

has the same value for all surfaces S that completely enclose the crack. Here $\mathbf{x}$ is the position

†This has been established rigorously for special cases (e.g. elliptic cracks,[8]), and has entered the lore of fracture mechanics for arbitrary cracks with smooth boundaries. Compelling arguments, omitted here, can easily be constructed. It should perhaps be noted in passing, however, that isotropy, or at least elastic symmetry with respect to the crack plane, is necessary for the validity of this asymptotic result.

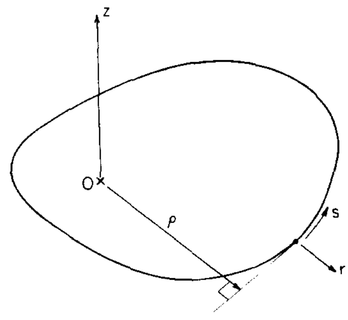

Fig. 2. Crack-based coordinates.

vector, **U** is the elastic displacement, **T** is the surface traction on S, **n** is the unit outward normal to S, W is the strain energy density, and ∇ is the gradient operator. Budiansky and Rice[10] interpreted this integral in terms of the energy release rate associated with self-similar growth of the crack, in which each point of C recedes radially from the origin at a rate proportional to its distance therefrom. Specifically, if $\mathscr{E}$ is the total energy released by such growth, then

$$a\frac{\partial \mathscr{E}}{\partial a} = M \tag{13}$$

where a, again, is a characteristic length that measures the crack size. Now put the origin in the crack surface, and choose S as shown in Fig. 3: two planes coincident with the crack faces, joined to a tunnel that surrounds the crack boundary C. In each plane normal to C let the tunnel cross-section be a circle l of radius δ. Then, since $\mathbf{x}\,.\,\mathbf{n} = \mathbf{T} = 0$ on the crack faces, the expression for M can be reduced to

$$M = \oint_C \lim_{\delta\to 0} \oint_l \{\ldots\}\,\mathrm{d}l\,\mathrm{d}s.$$

But **U** and $(\partial \mathbf{U}/\partial s)$ are bounded on C; $\mathbf{T} = 0(1/\sqrt{\delta})$: and **x** approaches the plane of the crack for $\delta \to 0$; hence M reduces further to

$$M = \oint_C \rho(s) \lim_{\delta\to 0} \oint_l \left(Wn_r - \mathbf{T}\cdot\frac{\partial \mathbf{U}}{\partial r}\right)\mathrm{d}l\,\mathrm{d}s \tag{14}$$

where (Fig. 2) $\rho(s)$ is the perpendicular distance from the origin to the tangent line to C at s, and n_r is the r-component of **n**. But the inner integral in (14) is precisely Rice's[11] well-known J integral of two-dimensional fracture mechanics. For any combination of plane strain and antiplane shear, J is given in terms of the stress-intensity factors by[12]

$$J = \frac{1-\nu^2}{E}[K_{\mathrm{I}}^2 + K_{\mathrm{II}}^2 + K_{\mathrm{III}}^2/(1-\nu)]. \tag{15}$$

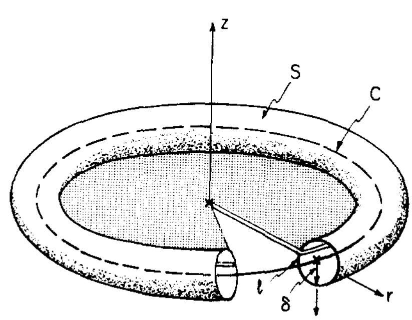

Fig. 3. Surface S surrounding crack C.

Hence, since plane strain and antiplane shear are approached for $\delta \to 0$, (13) and (14) give

$$a\frac{\partial \mathscr{E}}{\partial a} = \frac{1-\nu^2}{E}\oint_C \rho[K_I^2 + K_{II}^2 + K_{III}^2/(1-\nu)]\,\mathrm{d}s. \tag{16}$$

For the case of a crack in an infinite body, the K's must be proportional to $\sqrt{a}$. Hence, the total crack energy, found by integrating the energy released by growth of the crack from nothing up to its current size is

$$\mathscr{E} = \frac{1-\nu^2}{3E}\oint_C \rho[K_I^2 + K_{II}^2 + K_{III}^2/(1-\nu)]\,\mathrm{d}s. \tag{17}$$

A similar relation was given by Irwin[13] for elliptic cracks in mode I (i.e. $K_{II} = K_{III} = 0$). Irwin used the fact that (15) is the two-dimensional energy release rate associated with crack-tip extension and simply added up release rates associated with the normal motion of each part of the crack boundary in order to find the desired three-dimensional result. The present derivation, more elaborate and longer, is perhaps somewhat more convincing.

It is now possible, using (17), to calculate $\mathscr{E}$ for any crack in an infinite isotropic body for which the edge distributions of K_I, K_{II}, and K_{III} are known. These K factors are known for elliptic cracks[8], but a simple rederivation along the lines used by Irwin[13] for finding K_I will be shown next.†

ELLIPTIC CRACK ENERGIES

Consider an elliptic crack in the x-y plane (Fig. 4) having major and minor semi-axes a and b. In an infinite, homogeneous elastic body uniformly loaded at infinity the displacement jump $[\mathbf{U}] \equiv \mathbf{U}(x, y, 0^+) - \mathbf{U}(x, y, 0^-)$ across the crack faces will have the form

$$[\mathbf{U}] = \mathbf{A}(ab)^{1/2}[1 - x^2/a^2 - y^2/b^2]^{1/2} \tag{18}$$

where $\mathbf{A}$ is the constant vector $\mathbf{i}A + \mathbf{j}B + \mathbf{k}C$. This follows directly from the remarkable discovery by Eshelby[14] that a homogeneous ellipsoidal inclusion in an otherwise homogeneous, infinite elastic body uniformly loaded at infinity will suffer a uniform strain in its interior; (18) simply expresses the limit of this result for the case of an ellipsoidal void having semi-axes (a, b, c), for the case $c \to 0$. Furthermore, if the body is isotropic, applied stresses σ_x, σ_y, τ_{xy} produce $\mathbf{A} = 0$; and the external stresses σ_z, τ_{xz}, τ_{yz} produce non-zero values only for A, B, and C, respectively.

This result was evidently known to Irwin in[13] (from other studies) only for the case of tensile loading σ_z, and he used it to find K_I in the following way: Near a particular point $(\bar{x}, \bar{y})$ on the crack edge the jump $[w]$ has the form $[w] \approx c_z\sqrt{-r}$, and by two-dimensional plane-strain crack analysis (e.g.[11], p. 216), c_z is related to K_I by

$$K_I = \frac{(2\pi)^{1/2}Ec_z}{8(1-\nu^2)}. \tag{19}$$

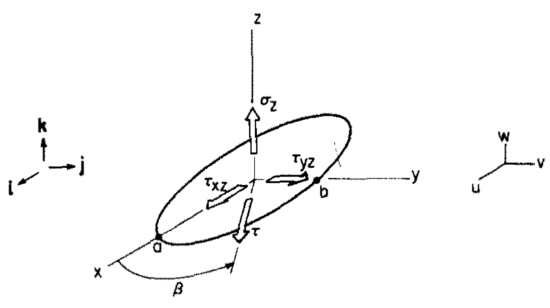

Fig. 4. Elliptic crack and resolved stresses.

†We are indebted to J. R. Rice for suggesting this extension of Irwin's method.

But with $[w] = A(ab)^{1/2}(1 - x^2/a^2 - y^2/b^2)^{1/2}$, $c_z = A(2ab)^{1/2}(\bar{x}^2/a^4 + \bar{y}^2/b^4)^{1/4}$, so that

$$K_{\rm I} = \frac{A(\pi ab)^{1/2}E}{4(1-\nu^2)}(\bar{x}^2/a^4 + \bar{y}^2/b^4)^{1/4}. \tag{20}$$

Consequently, $\mathscr{E}$ can be found in terms of A from (17). With the use of the substitutions $\bar{x} = a\cos\phi$, $\bar{y} = b\sin\phi$, we have $\rho\,\mathrm{d}s = ab\,\mathrm{d}\phi$, so that (17) leads to

$$\begin{aligned}\mathscr{E} &= \frac{\pi}{12}\frac{A^2Ea^2b}{(1-\nu^2)}\int_0^{\pi/2}[1-(1-b^2/a^2)\sin^2\phi]^{1/2}\,\mathrm{d}\phi \\ &= \left[\frac{\pi}{12}\frac{A^2Ea^2b}{(1-\nu^2)}\right]E(k)\end{aligned} \tag{21}$$

where $E(k)$ is the complete elliptic integral of the second kind, with argument $k = (1 - b^2/a^2)^{1/2}$.

But $\mathscr{E}$ can be found in another way, by noting that it must equal the work done by the applied stress acting (slowly) through the displacement of each crack face. Thus

$$\mathscr{E} = \frac{\sigma}{2}\int\int[w]\,\mathrm{d}x\,\mathrm{d}y = \frac{\pi\sigma(ab)^{3/2}A}{3}. \tag{22}$$

Equating (21) and (22) gives the value of A, from which $\mathscr{E}$ is found to be

$$\mathscr{E} = \frac{4\pi\sigma^2ab^2}{3E(k)}\left(\frac{1-\nu^2}{E}\right) \tag{23}$$

and, as a by-product, we also have the result

$$K_{\rm I} = \frac{\sigma\sqrt{\pi b/a}}{E(k)}(b^2\cos^2\phi + a^2\sin^2\phi)^{1/4} \tag{24}$$

found by Irwin. Finally, replacing E and ν by $\bar{E}$ and $\bar{\nu}$ in (23), and comparing with the first part of (8) gives

$$f(\bar{\nu}) = \left(\frac{4\pi}{3}\right)(b/a)^2\left(\frac{1-\bar{\nu}^2}{E(k)}\right). \tag{25}$$

There is now no difficulty in repeating this process for separate application of τ_{xz} and τ_{yz}, the effects of which, for elliptic cracks, are uncoupled. With only τ_{xz} present, we now have

$$[u] = B(ab)^{1/2}(1 - x^2/a^2 - y^2/b^2)^{1/2}$$

and this implies that near $(\bar{x}, \bar{y}) = (a\cos\phi, b\sin\phi)$ on the ellipse the jumps in the displacements in the r and s direction are $[u_r] = c_r\sqrt{-r}$ and $[u_s] = c_s\sqrt{-r}$, with

$$\begin{Bmatrix} c_r \\ c_s \end{Bmatrix} = \begin{Bmatrix} b\cos\phi \\ -a\sin\phi \end{Bmatrix} B(2)^{1/2}(b^2\cos^2\phi + a^2\sin^2\phi)^{-1/4}. \tag{26}$$

Again, from plane-strain results ([11], p. 216)

$$\left.\begin{aligned} K_{\rm II} &= \frac{(2\pi)^{1/2}Ec_r}{8(1-\nu^2)} \\ K_{\rm III} &= \frac{(2\pi)^{1/2}Ec_s}{4(1+\nu)} \end{aligned}\right\}. \tag{27}$$

Then (26), (27) and (17) lead to

$$\mathscr{E} = \frac{\pi}{12} \frac{B^2 E a^2 b}{(1-\nu^2)k^2} [(k^2 - \nu)E(k) + \nu k_1^2 K(k)] \tag{28}$$

where $K(k) = \int_0^{\pi/2} [1 - k^2 \sin^2 \phi]^{-1/2} \, d\phi$ is the complete elliptic integral of the first kind, and $k_1^2 = 1 - k^2$. The independent calculation of $\mathscr{E}$ is

$$\mathscr{E} = \frac{\tau_{xz}}{2} \int\int [u] \, dx \, dy = \frac{\pi \tau_{xz} (ab)^{3/2} B}{3} \tag{29}$$

and this, with (26)–(28) gives the desired results for $\mathscr{E}$, $K_{\rm II}$, $K_{\rm III}$. The corresponding answers for $\tau_{xz} = 0$, $\tau_{yz} \neq 0$ may then be written by inspection, and linear superposition, with $\tau_{xz} = \tau \cos \beta$, $\tau_{yz} = \tau \sin \beta$ gives

$$\mathscr{E} = \frac{4\pi\tau^2 ab^2}{3} \left(\frac{1-\nu^2}{E}\right) \{R(k, \nu) \cos^2 \beta + Q(k, \nu) \sin^2 \beta\} \tag{30}†$$

where

$$\begin{aligned} R(k, \nu) &= k^2[(k^2 - \nu)E(k) + \nu k_1^2 K(k)]^{-1} \\ Q(k, \nu) &= k^2[(k^2 + \nu k_1^2)E(k) - \nu k_1^2 K(k)]^{-1} \end{aligned} \tag{31}$$

Also, in agreement with Kassir and Sih[7],

$$K_{\rm II} = \frac{\tau \sqrt{\pi b / a} \, [bR(k, \nu) \cos \beta \cos \phi + aQ(k, \nu) \sin \beta \sin \phi]}{[b^2 \cos^2 \phi + a^2 \sin^2 \phi]^{1/4}} \tag{32}$$

$$K_{\rm III} = \frac{\tau(1-\nu)\sqrt{\pi b / a} \, [-aR(k, \nu) \cos \beta \sin \phi + bQ(k, \nu) \sin \beta \cos \phi]}{[b^2 \cos^2 \phi + a^2 \sin^2 \phi]^{1/4}}. \tag{33}$$

Comparison of (30) with (8) gives

$$g(\bar{\nu}, \beta) = \frac{4\pi}{3} \left(\frac{b}{a}\right)^2 (1 - \bar{\nu}^2)[R(k, \bar{\nu}) \cos^2 \beta + Q(k, \bar{\nu}) \sin^2 \beta]. \tag{34}$$

With this result for $g(\bar{\nu}, \beta)$, and eqn (25) for $f(\bar{\nu})$, we can proceed to the evaluation of (6) and (10) to get effective moduli of solids containing elliptic cracks.

EFFECTIVE MODULI (DRY CRACKS)

Under the simplifying assumption that all the cracks are elliptic and have the same aspect ratio b/a, $\langle f(\bar{\nu}) \rangle$ is given by (25). Consequently, substitution into (6) gives the results,

$$\bar{K}/K = 1 - \frac{8\pi N \langle ab^2 \rangle (1 - \bar{\nu}^2)}{9(1 - 2\bar{\nu})E(k)}. \tag{35}$$

But note that the area A of a crack is πab, and its perimeter is $P = 4aE(k)$, so that (35) can be rewritten in its final form

$$\bar{K}/K = 1 - \frac{16}{9} \left(\frac{1 - \bar{\nu}^2}{1 - 2\bar{\nu}}\right) \epsilon \tag{36}$$

†Except for a misprint, this result was essentially given by Eshelby[14, 15, eqn (6.7)]; the misprint was corrected by Eshelby in [16]. The result for the crack energy in the tension case (eqn (23)) has been shown correctly by Eshelby[15, eqn (6.6)].

where the crack-density parameter ϵ is defined by

$$\epsilon = \frac{2N}{\pi}\left\langle\frac{A^2}{P}\right\rangle. \tag{37}$$

With this choice for ϵ the result (36) does not depend explicitly on b/a; also ϵ reduces simply to $\epsilon = N\langle a^3\rangle$ in the case of circular cracks. Further, it is clear that the result (36), with the definition (37) for ϵ, continues to hold for bodies containing elliptical cracks of various b/a, as long as their size and aspect-ratio are uncorrelated.

The evaluation of $\langle g(\bar{\nu}, \beta)\rangle$ from (34) gives, for fixed b/a

$$\langle g(\bar{\nu}, \beta)\rangle = \frac{2\pi}{3}(b/a)^2(1-\bar{\nu}^2)[R(k, \bar{\nu}) + Q(k, \bar{\nu})] \tag{38}$$

since $\langle\cos^2\beta\rangle = \langle\sin^2\beta\rangle = (1/2)$. Substitution of $\langle f(\bar{\nu})\rangle$ and $\langle g(\bar{\nu}, \beta)\rangle$ into (10), with introduction of the definition (37) for ϵ, then gives

$$\bar{E}/E = 1 - \frac{16(1-\bar{\nu}^2)}{45}[3 + T(b/a, \bar{\nu})]\epsilon \tag{39}$$

where

$$\begin{aligned} T(b/a, \bar{\nu}) &= E(k)[R(k, \bar{\nu}) + Q(k, \bar{\nu})] \\ &= k^2E(k)\{[(k^2-\bar{\nu})E(k) + \bar{\nu}k_1{}^2K(k)]^{-1} + [(k^2+\bar{\nu}k_1{}^2)E(k) - \bar{\nu}k_1{}^2K(k)]^{-1}\} \\ &\qquad (k^2 = 1 - b^2/a^2 = 1 - k_1{}^2). \end{aligned} \tag{40}$$

The standard relation (5) among $\bar{K}$, $\bar{E}$, and $\bar{\nu}$, and the similar one for K, E, and ν, can be combined to

$$2(\nu - \bar{\nu}) = (1-2\bar{\nu})(1-\bar{K}/K) - (1-2\nu)(1-\bar{E}/E). \tag{41}$$

Substitution of (36) and (39) then provides the following relation among $\bar{\nu}$, ν and ϵ:

$$\epsilon = \frac{45}{8}\frac{\nu - \bar{\nu}}{(1-\bar{\nu}^2)[2(1+3\nu) - (1-2\nu)T]}. \tag{42}$$

Thus, for a given value of ν, the relation between $\bar{\nu}$ and ϵ is determined by (42), and then the variations of $\bar{K}/K$ and $\bar{E}/E$ with ϵ follow directly from (36) and (39). It is useful to note that $\bar{\nu}$ is a decreasing function of ϵ; that $T(b/a, 0) = 2$, which implies that for *all* values of b/a and ν, $\bar{\nu} \to 0$ for $\epsilon \to 9/16$; and consequently, again for all b/a and ν, $\bar{K}/K$ and $\bar{E}/E \to 0$ for $\epsilon \to 9/16$. This vanishing of the moduli can be interpreted as a loss of coherence of the material that is produced by an intersecting crack network at the critical value 9/16 of the crack density parameter. Although sufficient cracking will indeed have such an effect, it is clearly beyond the power of the present self-consistent computational approach to predict this critical condition with precision. Nevertheless, it is impressive that the method does predict a critical crack density, and it is plausible that very small, if not vanishing, stiffnesses will occur near $\epsilon = 9/16$.

To round out the results for $\bar{K}/K$ and $\bar{E}/E$, an analogous expression for $\bar{G}/G$ is easily found from standard elastic relations. Since $1/G - 3/E + 1/3K = 0$, and similarly for the barred quantities, it follows that

$$1 - \frac{\bar{G}}{G} - \frac{3\bar{G}}{\bar{E}}\left(1 - \frac{\bar{E}}{E}\right) - \frac{\bar{G}}{3\bar{K}}\left(1 - \frac{\bar{K}}{K}\right) = 0$$

or

$$2\left(1 - \frac{\bar{G}}{G}\right)(1+\bar{\nu}) - 3\left(1 - \frac{\bar{E}}{E}\right) + (1-2\bar{\nu})\left(1 - \frac{\bar{K}}{K}\right) = 0 \tag{43}$$

whence

$$\bar{G}/G = 1 - \frac{32}{45}(1-\bar{\nu})\left[1 + \frac{3}{4}T(b/a, \bar{\nu})\right]\epsilon. \tag{44}$$

An independent calculation of $\bar{G}/G$ by a direct procedure similar to that used for finding $\bar{E}/E$ gives the same answer.

For circular cracks ($b/a = 1$), $T = 4/(2-\bar{\nu})$, and $\epsilon = N\langle a^3\rangle$, and eqns (39), (44), and (42) reduce to

$$\bar{E}/E = 1 - \frac{16}{45}\frac{(1-\bar{\nu}^2)(10-3\bar{\nu})}{(2-\bar{\nu})}\epsilon \tag{39$'$}$$

$$\bar{G}/G = 1 - \frac{32}{45}\frac{(1-\bar{\nu})(5-\bar{\nu})}{(2-\bar{\nu})}\epsilon \tag{44$'$}$$

$$\epsilon = \frac{45}{16}\frac{(\nu-\bar{\nu})(2-\bar{\nu})}{(1-\bar{\nu}^2)[10\nu - \bar{\nu}(1+3\nu)]}. \tag{42$'$}$$

These relations, together with Eq. (36) for $\bar{K}/K$, give the results shown in Fig. 5, wherein $\bar{K}/K$, $\bar{E}/E$, $\bar{G}/G$, and $\bar{\nu}$ are plotted against $\epsilon = N\langle a^3\rangle$ for several values of ν.

For the limiting case of long narrow elliptic cracks ($b/a \to 0$), with $T = (2-\bar{\nu})/(1-\bar{\nu})$ and $\epsilon = (\pi/2)N\langle ab^2\rangle$ the results reduce to

$$\bar{E}/E = 1 - \frac{16}{45}(1+\bar{\nu})(5-4\bar{\nu})\epsilon \tag{39$''$}$$

$$\bar{G}/G = 1 - \frac{8}{45}(10-7\bar{\nu})\epsilon \tag{44$''$}$$

$$\epsilon = \frac{45}{8}\frac{\nu-\bar{\nu}}{(1+\bar{\nu})[10\nu - \bar{\nu}(1+8\nu)]} \tag{42$''$}$$

As already indicated, the moduli (and $\bar{\nu}$) still vanish at $\epsilon = 9/16$. Further, the detailed variations with ϵ of the moduli throughout the range $0 \le \epsilon \le 9/16$ turn out to be hardly different from those in Fig. 5. Indeed, as long as ϵ is defined by (37) the results of Fig. 5 are, to within a few percent, applicable to all values of b/a. This, in turn, means that the requirement of constant b/a for all cracks can be relaxed, as long as the elliptic crack orientations remain random, and uncorrelated with size and shape. In fact, it does not seem at all unlikely that the circular crack results may even be applicable, with little error, to all cracks of convex shape, again with the understanding that the general definition (37), in terms of the area and perimeter of the crack, holds for ϵ.

This supposition may be at least partially verified by calculating effective moduli for the case of very long rectangular cracks (length $2a$, width $2b$, $b/a \ll 1$). The crack energy, asymptotically

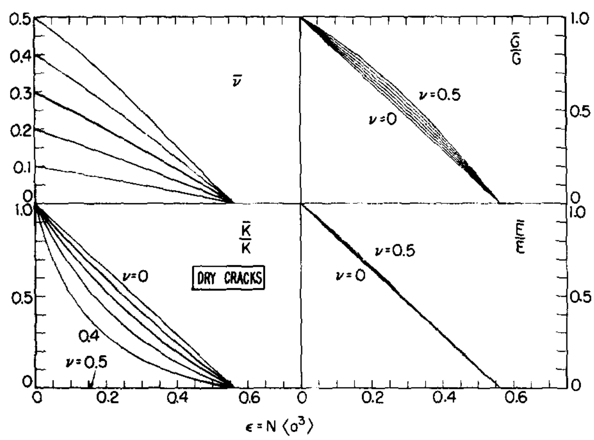

Fig. 5. Effective moduli; dry circular cracks.

for $b/a \to 0$, is easily found directly from the two-dimensional energy release rate (15), with the two-dimensional values $K_{\rm I} = \sigma\sqrt{\pi b}$, $K_{\rm II} = \tau \sin\beta\sqrt{\pi b}$, $K_{\rm III} = \tau\cos\beta\sqrt{\pi b}$. This gives

$$\mathscr{E} = \frac{2\pi ab^2}{\bar{E}}\left[\sigma^2 + \tau^2\sin^2\beta + \frac{\tau^2\cos^2\beta}{1-\bar{\nu}}\right][1-\bar{\nu}^2]$$

so that, by comparison with (8), $f(\bar{\nu})$ and $g(\bar{\nu})$ are determined. The results for the moduli, found from (6), (10), (41) and (43) are

$$\left.\begin{aligned} \bar{K}/K &= 1 - \frac{\pi^2}{6}\left(\frac{1-\bar{\nu}^2}{1-2\bar{\nu}}\right)\epsilon \\ \bar{E}/E &= 1 - \frac{\pi^2}{30}(1+\bar{\nu})(5-4\bar{\nu})\epsilon \\ \bar{G}/G &= 1 - \frac{\pi^2}{60}(10-7\bar{\nu})\epsilon \\ \epsilon &= \frac{60}{\pi^2}\frac{\nu-\bar{\nu}}{(1+\bar{\nu})[10\nu - \bar{\nu}(1+8\nu)]} \end{aligned}\right\} \tag{45}$$

where $\epsilon = (2/\pi)N\langle A^2/P\rangle = (8/\pi)N\langle ab^2\rangle$. In fact, these results for long rectangular cracks may be found from those for long elliptical cracks by replacing ϵ in eqns (36), (39)″, (44)″, and (42)″ by $(3\pi^2\epsilon/32)$. Thus, to inflict the same fractional reduction of moduli as do N long elliptical cracks of a given $\langle A^2/P\rangle$, about $(1{\cdot}08)N$ long rectangular cracks of the same $\langle A^2/P\rangle$ are needed. Note, however, that if the comparison is made on the basis of long ellipses and rectangles having the same a's and b's, 33% *fewer* rectangular cracks would suffice.

EFFECTIVE MODULI (SATURATED CRACKS)

Suppose the thin cracks contemplated contain a fluid of bulk modulus $\tilde{K}$. The mathematical assumption of a zero-thickness crack can no longer be invoked as casually as it was in the case of empty cracks. A small non-zero crack volume v_c must be assumed and then, as will be shown for ellipsoidal cracks, the new parameter

$$\omega \equiv \frac{8}{3}\left(\frac{A^2}{Pv_c}\right)\left(\frac{\tilde{K}}{K}\right) \tag{46}$$

enters the results in an essential way. For ellipsoids, $v_c = (4/3)\pi abc$, so that for thin oblate spheroidal cracks $(a = b \gg c)$, $\omega = (a/c)(\tilde{K}/K)$; long thin ellipsoidal cracks $(a \gg b \gg c)$ give $\omega = (\pi/2)(b/c)(\tilde{K}/K)$. The empty-crack case is recovered for $\omega = 0$, corresponding to $\tilde{K}\to 0$, $c \neq 0$; but note that if $\tilde{K} \neq 0$, a facile passage to the limit $c \to 0$ would give $\omega = \infty$. The right values of *both* (A^2/Pv_c) and $\tilde{K}/K$ must be used to determine which (if either) of these limits is appropriate. Consider, for example, air-filled cracks in rock, for which $\tilde{K}/K \sim 10^{-6}$; if the cracks are circular, with $c/a \sim 10^{-3}$, then $\omega \sim 10^{-3}$, and, as we shall see, this is close enough to zero. On the other hand, similar cracks filled with cool water $(\tilde{K}/K \approx 0{\cdot}03)$ give $\omega \approx 30$, and in this case the results for $\omega = \infty$ will be found satisfactory.

The results for arbitrary ω will now be derived. It is important to note here that a basic assumption in the calculations that follow is that the fluid in each crack is considered to be isolated. That is, the moduli to be found are appropriate for stress changes that occur with sufficient rapidity to prevent communication of fluid pressure *between* cracks. This is the situation that corresponds to elastic waves of sufficiently high frequency and is in contrast to other treatments that assume homogeneous fluid pressure throughout the body.

To do the calculation, it is only necessary to modify the expression (8) for the crack energy release, and this can be done quite generally for cracks of arbitrary shape, but vanishingly small thicknesses. The presence of fluid in the crack modifies the energy release in two ways. Due to the application of σ to the cracked body, the fluid itself will acquire some hydrostatic stress $\tilde{\sigma}$, and hence gain the strain energy

$$\frac{\tilde{\sigma}^2}{2\tilde{K}} v_c. \tag{47}$$

Also, the application of this stress $\tilde{\sigma}$ to the crack surfaces augments the energy of the surrounding body-load system by the amount

$$\frac{a^3 f(\bar{\nu})}{\bar{E}} \tilde{\sigma}^2. \tag{48}$$

(This result follows directly from eqn (2), which is the same as the work done by a hydrostatic pressure p slowly applied to the crack surface.) The quantities (47) and (48) must be *subtracted* from the energy release (8) for the dry crack, giving

$$\mathscr{E} = \frac{a^3 f(\bar{\nu})\sigma^2}{\bar{E}} \left\{1 - \left(\frac{\tilde{\sigma}}{\sigma}\right)^2 (1+\gamma)\right\} + \frac{a^3 g(\bar{\nu})\tau^2}{\bar{E}} \tag{49}$$

where

$$\gamma = \frac{3(1-2\bar{\nu})\bar{K} v_c}{2 f(\bar{\nu}) \tilde{K} a^3}. \tag{50}$$

Since the fluid can not carry shear stress, the contribution of τ to the energy is unaffected. To calculate $\tilde{\sigma}$, equate the volume change $(\tilde{\sigma}/\tilde{K})v_c$ of the fluid to the crack-volume change $(2a^3 f(\bar{\nu})/\bar{E})\,(\sigma - \tilde{\sigma})$; this gives

$$\tilde{\sigma}/\sigma = (1+\gamma)^{-1}$$

and then

$$\mathscr{E} = \frac{a^3}{\bar{E}} \{f(\bar{\nu}) D\sigma^2 + g(\bar{\nu})\tau^2\} \tag{51}$$

where

$$D = \frac{\gamma}{\gamma+1}. \tag{52}$$

It follows that the relations (6) and (10) for the moduli of dry-cracked bodies can be transformed to results for saturated cracks simply by replacing $f(\bar{\nu})$ by $f(\bar{\nu})D$. In the case of elliptic cracks, the use in (50) and (52) of (25) for $f(\bar{\nu})$ leads to

$$D = \left[1 + \frac{4}{3\pi}\left(\frac{K}{\tilde{K}}\right)\left(\frac{1-\bar{\nu}^2}{1-2\bar{\nu}}\right)\omega\right]^{-1} \tag{53}$$

where ω is defined by (46), and then (6) and (10) give

$$\bar{K}/K = 1 - \frac{16}{9}\left(\frac{1-\bar{\nu}^2}{1-2\bar{\nu}}\right) D\epsilon \tag{54}$$

$$\bar{E}/E = 1 - \frac{16(1-\bar{\nu}^2)}{45}[3D + T]\epsilon. \tag{55}$$

Finally, (43) and (41) give us

$$\bar{G}/G = 1 - \frac{32}{45}(1-\bar{\nu})\left[D + \frac{3}{4}T\right]\epsilon \tag{56}$$

and

$$\epsilon = \frac{45}{8} \frac{\nu - \bar{\nu}}{(1-\bar{\nu}^2)[2D(1+3\nu) - (1-2\nu)T]} \tag{57}$$

Note that D depends on $\bar{K}/K$, as well as $\bar{\nu}$ and ω, so that for given ϵ and ω, eqns (53), (54) and (57) must be solved simultaneously for D, $\bar{K}/K$, and $\bar{\nu}$.

The derivation of these results was approximate in the sense that the crack opening was imagined to be so small that the response of the material to pressures on the crack surface was assumed to be the same as that for perfectly flat cracks. In addition, no account was taken in the energy budget of the small strain energy associated with the solid material missing from the crack cavity. But if we consider the self-consistent moduli to be functions of ϵ, ω, and the "thickness" measure (Pv_c/A^2), the present results for these materials are exactly valid, in an asymptotic sense, in the limit $(Pv_c/A^2)\rightarrow 0$, with ϵ and ω fixed. Perhaps a better way to discuss this is to introduce the crack porosity $\eta = Nv_c$, so that

$$\omega = \frac{4\pi}{3}\left(\frac{\tilde{K}}{K}\right)\left(\frac{\epsilon}{\eta}\right). \tag{58}$$

The actual self-consistent moduli are functions $F(\omega, \epsilon, \eta)$, and we have found, without approximation, $\lim_{\eta\rightarrow 0} F(\omega, \epsilon, \eta)$. Accordingly, the results, for given ω and ϵ, should be reliable (within the limitations of the self-consistent method) for a sufficiently small crack porosity η.

The special case $\omega = \infty$ $(D = 0)$ gives the results shown in Fig. 6 for circles $(b/a = 1;$ $T = 4/[2-\nu])$. As in the dry-crack case, there is a critical value of ϵ for which $\bar{E}$ (and $\bar{G}$) vanish. However, in contrast to the dry-crack situation, the critical value of ϵ depends on the b/a ratio; for $\omega = \infty$ it varies between 45/32 for circular cracks and 5/4 for long elliptical cracks. A typical comparison of various results for wet and dry cracks of various shapes is given, for $\nu = 1/4$, in Fig. 7. As in the case of dry cracks, the wet-crack results for long rectangular cracks are those for

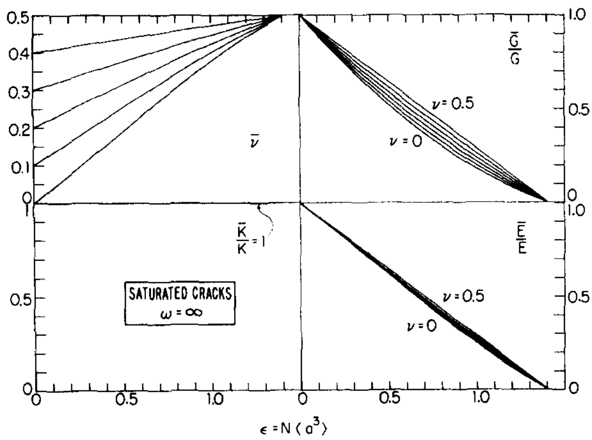

Fig. 6. Effective moduli; circular cracks saturated with "hard" fluid, $\omega \equiv (a/c)(\tilde{K}/K) = \infty$.

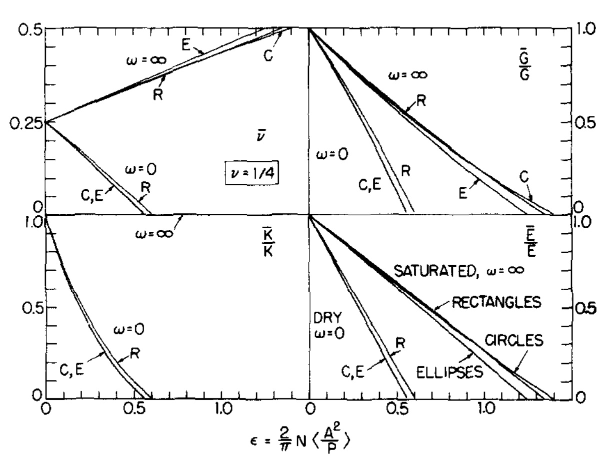

Fig. 7. Comparison of effective moduli: circular cracks (C); long elliptic cracks, $b/a = 0$ (E); and long rectangular cracks, $b/a = 0(R)$.

long elliptical cracks with ϵ replaced by $(3\pi^2/32)\epsilon$. Although the critical wet-crack ϵ's are not too far apart from each other, they differ substantially from those for dry cracks, which are 9/16 for all ellipses, and $6/\pi^2$ for long rectangles. This is not inconsistent with the view that cracks saturated with a "hard" fluid ($\omega = \infty$) should permit a larger crack density prior to loss of elastic coherence. Note also that for wet cracks (with $\omega = \infty$) Poisson's ratio always increases to the limiting value $\bar{\nu} = 1/2$, in contrast to the reduction to zero in the dry case.

The previously mentioned close agreement between the elliptic-crack results for $b/a = 1$ and 0 in the dry-crack case at all intermediate values of ϵ is so close for $\nu = 1/4$ as to render the corresponding curves in Fig. 7 indistinguishable, and close to the curve for long rectangular cracks. This, together with the fair agreement among circles, long ellipses, and long rectangles when $\omega = \infty$ makes it plausible that the results (53)–(57) will remain at least approximately valid for arbitrary convex cracks and all ω, as long as ω is given by the general definition (46). Additional numerical results will be displayed only for circular cracks, as in Fig. 8, which show the moduli for $\nu = 1/4$ and various values of ω. Here we note that the dry case ($\omega = 0$) is very well approximated for $\omega < 0{\cdot}01$; that $\omega = \infty$ may be used with little error for $\bar{E}$ and $\bar{G}$ whenever $\omega > 10$ (with a somewhat larger error in $\bar{K}$ until we get to $\omega > 100$). Note, too, the interesting non-uniform variation of $\bar{\nu}$ with ω; $\bar{\nu}$ always approaches 1/2, with a sharp turn-around near $\epsilon = 9/16$ for small ω.

PARTIAL SATURATION

It is easy to generalize the preceding analysis to include the situation in which only a fraction ξ of the cracks are saturated. This is done by using for the basic crack energy release a simple weighted average of the expressions (8) and (51), giving

$$\mathscr{E} = \frac{a^3}{E}\{(1-\xi+\xi D)f(\bar{\nu})\sigma^2 + g(\bar{\nu})\tau^2\} \tag{59}$$

The consequence is simply that the results for partially saturated bodies containing elliptic cracks are still given by eqns (54)–(57) with D everywhere replaced by $(1-\xi+\xi D)$. (The definition for D itself, eqn (53), remains unchanged.) Indeed, we can generalize further and consider a set of partial saturation fractions ξ_i, each corresponding to a random distribution of cracks having fluid parameter ω_i. With dry cracks ($\omega = 0$) included in this set we have $\Sigma\xi_i = 1$, and then the results (54)–(57) hold with D replaced by $\Sigma\xi_i D_i$, where D_i is given in terms of ω_i by (53).

Illustrative numerical results for $\nu = 1/4$ are shown in Fig. 9, for circular cracks, and "hard" liquid ($\omega = \infty$, $D = 0$) partial saturation to various degrees ξ. Here the critical values of ϵ depend on ξ, as do the limiting values of $\bar{\nu}$.

An interesting sidelight on eqns (54)–(57) and their generalized interpretations for partial

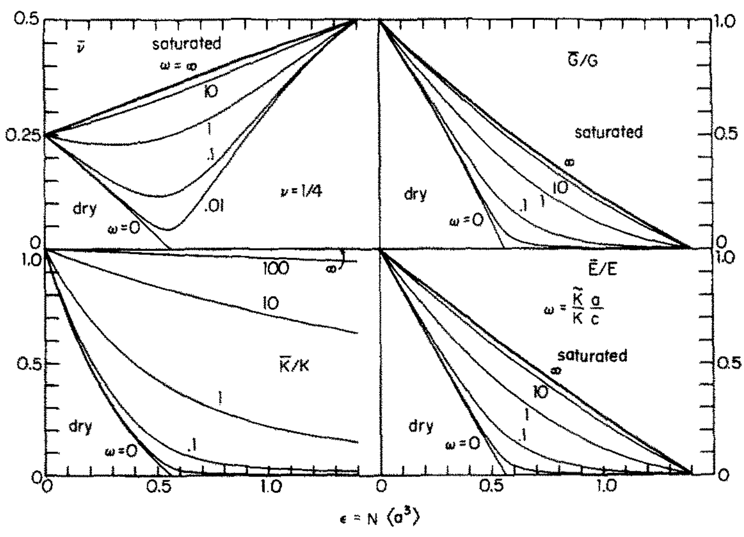

Fig. 8. Effective moduli; results for circular cracks saturated with liquid, hardness parameter ω.

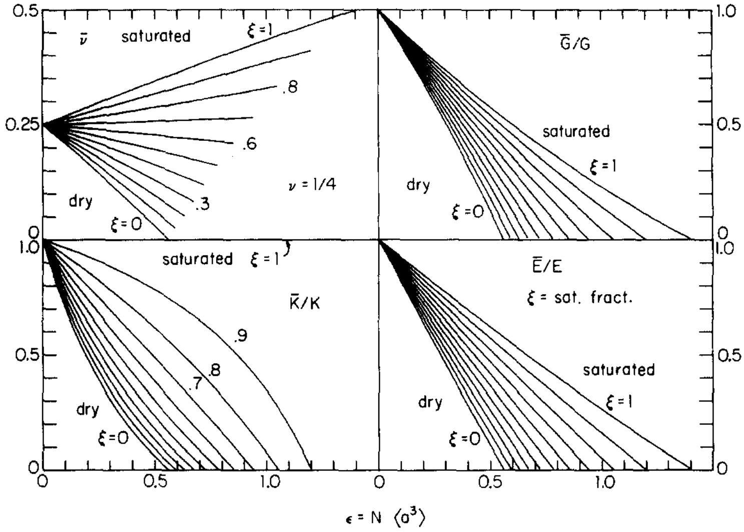

Fig. 9. Effective moduli; results for circular cracks, partially saturated with hard fluid, $\omega = \infty$.

saturation is that measurements of $\bar{\nu}$ and $\bar{K}/K$ (or $\bar{\nu}$ and any other effective modulus ratio like $\bar{G}/G$), can serve as a theoretical diagnostic for *ϵ that is independent of ξ and ω*. By eliminating D between (54) and (57) we find the relation

$$\epsilon = \frac{9(1-2\bar{\nu})(1+3\nu)(1-\bar{K}/K) - 45(\nu-\bar{\nu})}{8(1-\bar{\nu}^2)(1+2\nu)T}. \tag{60}$$

Similarly, the use of (56) gives

$$\epsilon = \frac{9(1+\bar{\nu})(1+3\nu)(1-\bar{G}/G) - 18(\nu-\bar{\nu})}{8(1-\bar{\nu}^2)(1+\nu)T}. \tag{61}$$

It is apparent that these relations also continue to apply for any arbitrary distribution of ω in the cracks, as long as macroscopic isotropy is not violated. Note that the variation of T between $4/(2-\bar{\nu})$ and $(2-\bar{\nu})/(1-\bar{\nu})$ for b/a between unity and zero constitutes a 12% variation in the extreme case $\bar{\nu} = 1/2$, and only 2% for $\bar{\nu} = 1/4$. Barring extremely elongated ellipses and near-critical conditions of full saturation the use of $T = 4/(2-\bar{\nu})$ in (60) or (61) should be quite accurate for all convex cracks. With this choice, (61) can be transformed into another form of possible practical use by means of the standard relations

$$\bar{V}_s/V_s = (\bar{G}/G)^{1/2}$$

$$V_p/V_s = \left[\frac{2(1-\nu)}{1-2\nu}\right]^{1/2}$$

$$\bar{V}_p/\bar{V}_s = \left[\frac{2(1-\bar{\nu})}{1-2\bar{\nu}}\right]^{1/2}$$

for the P and S wave velocities V_p and V_s in the uncracked body, and their counterparts $\bar{V}_p$, $\bar{V}_s$ in the cracked body. This gives

$$\epsilon = \frac{9\{[1-\bar{V}_s/V_s)^2][3(\bar{V}_p/\bar{V}_s)^2-4][5(V_p/V_s)^2-8] - 4[(V_p/V_s)^2-(\bar{V}_p/\bar{V}_s)^2]\}\{3(\bar{V}_p/\bar{V}_s)^{-}-2\}}{32(\bar{V}_p/\bar{V}_s)^2[3(\bar{V}_p/\bar{V}_s)^2-4][3(V_p/V_s)^2-4]}. \tag{62}$$

It is worth emphasizing that this formula giving the crack-density parameter in terms of V_p, V_s, $\bar{V}_p$, and $\bar{V}_s$ is supposed to hold under all conditions of full or partial crack saturation, and variable crack thickness.

GEOMETRICAL ESTIMATES OF CRACK DENSITY

We conclude with the presentation of two results of a purely geometrical nature, derived in the Appendix, for the estimation of ϵ on the basis of observations of the traces of cracks on a plane cross-section of the cracked body. On such a cross-section we can (in principle) measure various moments of the distribution $m(l)$ per unit area of the lengths l of discrete traces of flat cracks. Thus

$$\left.\begin{aligned}\langle l\rangle &= \frac{1}{M}\int_0^\infty l m(l)\,\mathrm{d}l \\ \langle l^2\rangle &= \frac{1}{M}\int_0^\infty l^2 m(l)\,\mathrm{d}l\end{aligned}\right\} \tag{63}$$

where

$$M = \int_0^\infty m(l)\,\mathrm{d}l$$

is the total number of traces per unit area. Remarkably, it is then true that for a random distribution of cracks of *any* given convex shape, all of the same size,

$$\epsilon = \frac{8}{\pi^3} M\langle l\rangle^2 \tag{64}$$

where ϵ is given by the general definition (37).

The restriction to uniform crack size may make this result of limited utility. However, if we consider elliptic cracks of variable size, but uniform aspect ratio b/a, then ϵ can be related to $\langle l^2\rangle$ by

$$\epsilon = \frac{3\pi}{16E(k)K(k)} M\langle l^2\rangle \tag{65}$$

The numerical factor varies by less than 10% for $4 < b/a < 1$, but then rapidly approaches zero for $b/a \to 0$.

REFERENCES

1. J. R. Bristow, Microcracks, and the static and dynamic elastic constants of annealed and heavily cold-worked metals. *Brit. J. Appl. Physics* **11**, 81 (1960).
2. J. B. Walsh, New analysis of attenuation in partially melted rock. *J. Geophys. Res.* **74**, 4333 (1969).
3. H. D. Garbin and L. Knopoff, The compressional modulus of a material permeated by a random distribution of circular cracks. *Quart. Appl. Math.* **30**, 453 (1973).
4. R. Hill, A self-consistent mechanics of composite materials. *J. Mech. Phys. Solids* **13**, 213 (1965).
5. B. Budiansky, On the elastic moduli of some heterogeneous materials. *J. Mech. Phys. Solids* **13**, 223 (1965).
6. B. Budiansky, Thermal and thermoelastic properties of isotropic composites. *J. Composite Materials* **4**, 286 (1970).
7. R. J. O'Connell and B. Budiansky, Seismic velocities in dry and saturated cracked solids. *J. Geophys. Res.* **79**, 5412 (1974).
8. M. K. Kassir and G. C. Sih, Three dimensional stress distribution around an elliptic crack under arbitrary loadings. *J. Appl. Mech.* (Sept. 1966).
9. J. K. Knowles and E. Sternberg, On a class of conservation laws in linearized and finite elastostatics. *Archiv. Rat. Mech. Anal.* **44**, 3 (1972).
10. B. Budiansky and J. R. Rice, Conservation laws and energy-release rates, *J. Appl. Mech.* (March 1973).
11. J. R. Rice, A path independent integral and the approximate analysis of strain concentration by notches and cracks. *J. Appl. Mech.* (June 1967).
12. J. R. Rice, Mathematical analysis in the mechanics of fracture. In *Fracture* (Edited by H. Liebowitz). Vol. II, Chap. 3. Academic Press, New York (1968).
13. G. R. Irwin, Crack-extension force for a part-through crack in a plate, *J. Appl. Mech.* (Dec. 1962).
14. J. D. Eshelby, The determination of the elastic field of an ellipsoidal inclusion, and related problems. *Proc. Roy. Soc.* **A241**, 376 (1957).
15. J. D. Eshelby, Elastic inclusions and inhomogeneities. In *Progress in Solid Mechanics*, (Edited by Sneddon and Hill), Vol. II, Chap. III. North-Holland, Amsterdam (1961).
16. J. D. Eshelby, The distribution of dislocations in an elliptical glide zone. *Phys. Stat. Sol.* **3**, 2057 (1963).
17. M. C. Kendall and P. A. P. Moran, *Geometrical Probability*. Charles Griffin, London (1963).
18. R. L. Salganik, Mechanics of bodies with many cracks. *Izv. AN SSSR, Mekhanika Tverdogo Tela* Vol. 8, No. 4, pp. 149–158 (1973). (English translation, Mechanics of Solids, pp. 135–143).

APPENDIX

Geometrical estimates of ϵ

The calculations below were stimulated by the monograph [17], in which the differential measure of a continuum of planes is defined by

$$\mathrm{d}\mathcal{N} = \mathrm{d}\Omega\, \mathrm{d}r \tag{A1}$$

where $\mathrm{d}\Omega$ is a solid angle containing plane normals, and $\mathrm{d}r$ is an incremental normal displacement of a plane. Then the measure of all planes intersecting a convex plane curve C (Fig. 10) is

$$\mathcal{N} = \int_H \mathrm{d}\Omega \int_{-\rho_1 \sin\theta}^{\rho_2 \sin\theta} \mathrm{d}r$$

where $\mathrm{d}\Omega = \sin\theta\ \mathrm{d}\theta\ \mathrm{d}\beta$, and H is the Northern hemisphere. Hence

$$\begin{aligned}\mathcal{N} &= \int_0^{2\pi} (\rho_1 + \rho_2)\,\mathrm{d}\beta \int_0^{\pi/2} \sin^2\theta\ \mathrm{d}\theta \\ &= \frac{\pi P}{2}\end{aligned} \tag{A2}$$

where P is the perimeter of C.

It follows that the average "thickness" of C—that is, the average over all orientations of the distance through which normal displacement of a plane keeps it intersecting C—must be

$$\langle t \rangle = \frac{\mathcal{N}}{2\pi} = \frac{P}{4}. \tag{A3}$$

Consider now N randomly distributed cracks per unit volume, each of the same size and shape. A plane slice through the body will intersect $M = N\langle t \rangle$ of these cracks per unit area; hence

$$M = \frac{NP}{4}. \tag{A4}$$

The average length of the crack traces in such a cross-section is ([17], p. 79)

$$\langle l \rangle = \frac{1}{\mathcal{N}} \int_H \mathrm{d}\Omega \int_{-\rho_1 \sin\theta}^{\rho_2 \sin\theta} l\ \mathrm{d}r.$$

With the variable change $r = \tilde{\rho} \sin\theta$, this is

$$\langle l \rangle = \frac{1}{\mathcal{N}} \int_0^{2\pi} \mathrm{d}\beta \int_0^{\pi/2} \sin^2\theta\ \mathrm{d}\theta \int_{-\rho_1}^{\rho_2} l\ \mathrm{d}\tilde{\rho}$$

and, since $\int_{-\rho_1}^{\rho_2} l\ \mathrm{d}\tilde{\rho} = A$, the crack area, the result is

$$\langle l \rangle = \frac{\pi A}{P}. \tag{A5}$$

(This calculation is attributed in [17] to Barbier, 1860.)

With the somewhat restrictive assumption, then, of uniform crack size and shape, (A4) and (A5) combine to give eqn (65) for ϵ.

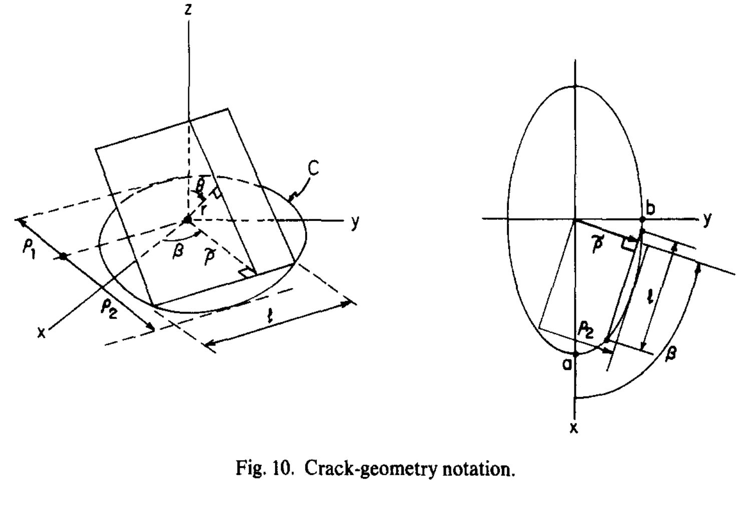

Fig. 10. Crack-geometry notation.

It also follows that

$$M\langle l\rangle = \frac{\pi}{4} NA$$

and this relation generalizes nicely when there are several species of cracks, each randomly distributed. Thus, the average trace length of the ith species is

$$\langle l_i \rangle = \frac{\pi A_i}{P_i}$$

and, with M_i and N_i the area and volume densities associated with the traces and cracks of the ith species,

$$\sum M_i \langle l_i \rangle = \pi \sum \frac{M_i A_i}{P_i} = \frac{\pi}{4} \sum N_i A_i.$$

Hence

$$M\langle l\rangle = \frac{\pi}{4} N\langle A\rangle \tag{A6}$$

an interesting, if not obviously useful, result.

To seek an estimate for ϵ that is not restricted to uniform crack size, we compute the mean square trace length, as follows:

$$\begin{aligned}\langle l^2\rangle &= \frac{1}{\mathcal{N}} \int_H \mathrm{d}\Omega \int_{-\rho_1 \sin\theta}^{\rho_2 \sin\theta} l^2 \, \mathrm{d}r \\ &= \frac{1}{\mathcal{N}} \int_0^{2\pi} \mathrm{d}\beta \int_0^{\pi/2} \sin^2\theta \, \mathrm{d}\theta \int_{-\rho_1}^{\rho_2} l^2 \, \mathrm{d}\tilde{\rho}\end{aligned}$$

and so

$$\langle l^2\rangle = \frac{1}{2P} \int_0^{2\pi} \mathrm{d}\beta \int_{-\rho_1}^{\rho_2} l^2 \, \mathrm{d}\tilde{\rho} \tag{A7}$$

We can write

$$\langle l^2\rangle = \left(\frac{A}{P}\right)^2 \psi \tag{A8}$$

where

$$\psi \equiv \frac{P}{2A^2} \int_0^{2\pi} \mathrm{d}\beta \int_{-\rho_1}^{\rho_2} l^2 \, \mathrm{d}\tilde{\rho} \tag{A9}$$

is a non-dimensional factor that depends only on the crack shape. In the presence of multiple crack species, this time each of the *same* shape but different sizes,

$$\sum M_i \langle l_i^2\rangle = \psi \sum M_i \left(\frac{A_i}{P_i}\right)^2 = \frac{\psi}{4} \sum \frac{N_i A_i^2}{P_i}$$

Hence

$$M\langle l^2\rangle = \frac{\psi}{4} N \left\langle \frac{A^2}{P} \right\rangle \tag{A10}$$

and $\epsilon = (2N/\pi)\langle A^2/P\rangle$ becomes

$$\epsilon = \frac{8}{\pi\psi} M\langle l^2\rangle. \tag{A11}$$

The evaluation of ψ for ellipses is straightforward. If the normal to l makes an angle β (see Fig. 10) with the major axis, then l^2 is found to be

$$l^2 = \frac{4a^2 b^2 [\rho_2^2 - \tilde{\rho}^2]}{\rho_2^4}$$

where $\tilde{\rho}$ is measured from the center of the ellipse, and

$$\rho_2^2 = \rho_1^2 = a^2 \cos^2\beta + b^2 \sin^2\beta$$

Then $\int_{-\rho_1}^{\rho_2} l^2 \, \mathrm{d}\tilde{\rho} = (16a^2 b^2/\rho_2)$, and (A9) leads directly to

$$\psi = \frac{128}{3\pi^2} E(k) K(k) \tag{A12}$$

Substitution into (A11) gives the final result, eqn (66), for ϵ in terms of $M\langle l^2\rangle$, for elliptic cracks.